Collective motion of self-propelled particles: kinetic phase transition in one dimension

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We demonstrate that a system of self-propelled particles (SPP) exhibits spontaneous symmetry breaking and self-organization in one dimension, in contrast with previous analytical predictions. To explain this surprising result we derive a new continuum theory that can account for the development of the symmetry broken state and belongs to the same universality class as the discrete SPP model.

The transport properties of systems consisting of self-propelled particles (SPP) have generated much attention lately [1–6]. This interest has been largely motivated by analogous processes taking place in numerous biological phenomena (e.g., bacterial migration on surfaces [7], flocking of birds, fish, quadrupeds [8], correlated motion of ants [9] and pedestrians [10]) as well as in various other systems, including driven granular materials [11,12] and traffic models [13]. The models describing these phenomena are distinctively non-equilibrium, exhibiting kinetic phase transitions and self-organization, of particular interest from the point of view of modern statistical mechanics [14].

In the simplest version of the SPP model [1] – introduced to study collective biological motion – each particle's velocity is set to a fixed magnitude, v_0. The interaction with the neighboring particles changes only the direction of motion: the particles tend to align their orientation to the local average velocity. Numerical simulations in 2D provided evidence of a second order phase transition [15] between an ordered phase in which the mean velocity of the entire system, $\langle v\rangle$, is nonzero and a disordered phase with $\langle v\rangle = 0$, as the strength of the noise is increased or the density of the particles decreased.

This SPP model is similar to the XY model of classical magnetic spins because the velocity of the particles, like the local spin of the XY model, has fixed length and continuous rotational symmetry. In the $v_0 = 0$ and low noise limit the model reduces *exactly* to a Monte-Carlo dynamics of the XY model. Since the XY model does *not* exhibit a long-range ordered phase at temperatures $T > 0$ [16], the ordered state observed in [1] is surprising. To explain this discrepancy, Toner and Tu (TT) [3] proposed a continuum theory that included in a self-consistent way the non-equilibrium effects as well. They have shown that their model is different from the XY model for $d < 4$ and found an ordered phase in $d = 2$ [17]. While TT could account for the first time for the ordered phase in 2D SPP models, their theory does not allow for an ordered phase for $d = 1$.

In this paper we demonstrate that kinetic phase transition and ordering takes place in 1D as well. This result, not foreseen by the existing analytical approaches, motivated us to introduce a new continuum theory describing the SPP model in arbitrary dimensions. Linear stability analysis shows that indeed in 1D the continuum model exhibits ordering. Furthermore, numerical investigations indicate that the continuum theory and the discrete SPP model belong to the same universality class.

The 1D SPP model — Let us consider N off-lattice particles along a line of length L. The particles are characterized by their coordinate x_i and dimensionless velocity u_i updated as

$$\begin{aligned} x_i(t+1) &= x_i(t) + v_0 u_i(t), \\ u_i(t+1) &= G\Big(\langle u(t)\rangle_i\Big) + \xi_i. \end{aligned} \tag{1}$$

The local average velocity $\langle u\rangle_i$ for the ith particle is calculated over the particles located in the interval $[x_i - \Delta, x_i + \Delta]$, where we fix $\Delta = 1$. The function G incorporates both the propulsion and friction forces which set the velocity in average to a prescribed value v_0: $G(u) > u$ for $u < 1$ and $G(u) < u$ for $u > 1$ [18]. The distribution function $P(x = \xi)$ of the noise ξ_i is uniform in the interval $[-\eta/2, \eta/2]$.

Keeping v_0 constant ($v_0 = 0.1$), the adjustable control parameters of the model are the average density of the particles, $\rho = N/L$, and the noise amplitude η. We implemented one of the simplest choices [19] for G,

$$G(u) = \begin{cases} (u+1)/2 & \text{for } u > 0 \\ (u-1)/2 & \text{for } u < 0, \end{cases} \tag{2}$$

and applied random initial and periodic boundary conditions.

In Fig. 1 we show the time evolution of the model for $\eta = 2.0$. In a short time the system reaches an ordered state, characterized by a spontaneous broken symmetry

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and clustering of the particles. In contrast, for larger values of η a disordered velocity field can be found.

Scaling and exponents — To capture quantitatively the transition from an ordered to a disordered state, in Fig. 2a we plot the order parameter $\phi \equiv \langle u \rangle$ vs η for various ρ. As in the two dimensions [1,15], the ordered phase emerges through a second order phase transition. Near the critical noise amplitude, $\eta_c(\rho, L)$, which separates the ordered from the disordered phase, ϕ vanishes as

$$\phi(\eta,\rho) \sim \begin{cases} \left(\frac{\eta_c(\rho,L)-\eta}{\eta_c(\rho,L)}\right)^{\beta} & \text{for } \eta < \eta_c(\rho,L) \\ 0 & \text{for } \eta > \eta_c(\rho,L) \end{cases}, \qquad (3)$$

which finding is supported by the increasing scaling regime with the system size L (Fig. 2b) and by the convergence of $\eta_c(\rho, L)$ to a non-zero $\eta_c(\rho, \infty)$ value. We find that $\beta = 0.60 \pm 0.05$, which is different from both the mean-field value $1/2$ [20] and $\beta = 0.42 \pm 0.03$ found in $d = 2$ [15].

Fig. 2a also shows that the various $\phi(\eta, \rho)$ curves can be collapsed onto a single function $\phi_0(x)$, where $x = \eta/\eta_c(\rho)$, just like in $d = 2$. As shown in [15], the consequence of this fact is that near the critical density the order parameter vanishes as

$$\phi(\eta,\rho) \sim \begin{cases} \left(\frac{\rho-\rho_c(\eta,L)}{\rho_c(\eta,L)}\right)^{\beta'} & \text{for } \rho > \rho_c(\eta,L) \\ 0 & \text{for } \rho < \rho_c(\eta,L) \end{cases}, \qquad (4)$$

with $\beta' = \beta$. These results can be summarized in the $\rho - \eta$ phase diagram shown in Fig. 2c. We also find that the critical line, $\eta_c(\rho)$, follows

$$\eta_c(\rho) \sim \rho^{\kappa}, \qquad (5)$$

with $\kappa = 0.25 \pm 0.05$.

While the above numerical results demonstrate the existence of the phase transition in one dimension and provide numerical values for the scaling exponents β, β' and κ, the origin of these values is unclear at this point. In particular, the emergence of the ordered phase in 1D is not predicted either by the equilibrium theories or by the TT model. To overcome this limitation next we introduce and investigate a set of continuum equations (which can be generalized to any dimension) in terms of $U(x, t)$ and $\varrho(x, t)$, where U and ϱ represent the coarse-grained dimensionless velocity and density fields, respectively. Analogous approaches were fruitful in the study of a similar SPP system, one-lane traffic flow [21].

Continuum theory — Let us denote by $n(u, x, t)dudx$ the number of particles moving with a velocity in the range of $[v_0 u, v_0(u + du)]$ at time t in the $[x, x + dx]$ interval. The particle density $\varrho(x, t)$ is then given as $\varrho = \int n du$, while the local dimensionless average velocity $U(x, t)$ can be calculated as $\varrho U = \int n u du$. According to the microscopic rules of the dynamics, in a given time interval $[t, t + \tau]$ all particles choose a certain velocity $v/v_0 = [G(\langle u \rangle) + \xi]$ and travel a distance $v\tau$. Thus, the time development of the ensemble average (denoted by overline) of n is governed by the master equation $\overline{n(u, x, t + \tau)} = \overline{\varrho(x', t)} p(u|U(x', t))$, where $x' = x - v_0 u\tau$ and $p(u|U)$ denotes the conditional probability of finding a particle with a velocity u when the local velocity field U is given. From Eq. (1) we have $p(u|U) = P(u - G(\langle U \rangle))$. Since n is finite, the actual occupation numbers in a given system differ from $\overline{n}$. This fact can be accounted by adding an intrinsic noise term to the master equation as

$$n(u, x, t + \tau) = \varrho(x', t) p(u|U(x', t)) + \nu(u, x', t), \qquad (6)$$

where ν has the following properties: (i) $\overline{\nu} = 0$, (ii) due to the conservation of the particles $\int \nu du = 0$, and (iii) since we have a random sampling process, the actual values of n satisfy Poisson statistics, i.e., the distribution function of ν depends on ϱ, u and U as $P(\nu) = \lambda^{\nu+\lambda} \exp(-\lambda)/\Gamma(\nu + \lambda + 1)$, where $\lambda = \varrho p(u|U)$. Thus, we have $\overline{\nu^2} = \overline{n}$.

Taking the Taylor expansion of $n(u, x - v_0 u\tau, t)$ up to second order in x and integrating Eq. (6) according to du, in the $v_0\tau \ll 1$, $\sigma^2 \equiv \int P(u)u^2 du \gg 1$, $\varrho \gg 1$ and $v_0\tau\sigma^2 < 1$ limit we obtain

$$\partial_t \varrho = -v_0 \partial_x(\varrho u) + D\partial_x^2 \varrho, \qquad (7)$$

where $D = v_0^2 \tau\sigma^2/2$. Note, that the appearance of the diffusion term is a consequence of the non-vanishing correlation time τ. Since $\int p(u|U) u du = G(\langle U \rangle)$, integrating Eq.(6) according to udu, expanding $\langle U \rangle$ as $\langle U \rangle = U + [\partial_x^2 U + 2(\partial_x U)(\partial_x \varrho)/\varrho]/6$ [22], using (7), we arrive at

$$\partial_t U = f(U) + \mu^2 \partial_x^2 U + \alpha \frac{(\partial_x U)(\partial_x \varrho)}{\varrho} + \zeta, \qquad (8)$$

where $f(U) = (G(U) - U)/\tau$, $\mu^2 = (dG/dU)/(6\tau)$, $\alpha = 2\mu^2$ and $\zeta = \int \nu u du/\varrho\tau$. Note, that $f(U)$ is an antisymmetric function with $f(U) > 0$ for $0 < U < 1$ and $f(U) < 0$ for $U > 1$, $\overline{\zeta} = 0$, and $\overline{\zeta^2} = \sigma^2/\varrho\tau^2$.

At this point we consider Eqs. (7) and (8) with the coefficiens μ, α, σ, v_0 and D as the continuum theory describing a large class of SPP models. These equations differ from both the equilibrium field theories and the nonequilibrium system investigated by TT [3]. The main differences are due to (i) the nonlinear coupling term $(\partial_x U)(\partial_x \varrho)/\varrho$, and (ii) the statistical properties of the noise ζ. For $\alpha = 0$ the dynamics of the velocity field U is independent of ϱ, and Eq.(8) is equivalent to the Φ^4 model describing spin chains, where domains of opposite magnetization develop at finite temperatures [20].

To study the effect of the nonlinear term in (8), we now investigate the development of the ordered phase in the deterministic case ($\sigma = 0$). For $\alpha = 0$ Eqs. (7) and (8) have a set of (meta)stable stationary solutions ϱ^* and U^* describing a "domain wall" separating two regions with opposite velocities. Since we can freely translate these

solutions, we assume $U^*(0) = 0$. Performing linear stability analysis we next show that for certain finite values of α the above stationary solutions are unstable.

Linear Stability Analysis — In the following we make use of the fact that the dynamics of ϱ is very slow compared to that of U. Let us write U in the form of $U(x,t) = U_0(x,t) + u(x,t)$, where $U_0(x,t) = U^*(x - \xi(t))$ and $\xi(t)$ is defined by $U(\xi(t),t) = 0$, i.e. $\xi(t)$ defines the position of the domain wall. Moreover, we substitute $f = U - U^3$ as one of the simplest choices for $f(U)$. Now in the $u \ll U_0$ and $\partial_x u \ll \partial_x U_0$ limit in the moving frame $x' = x - \xi(t)$ Eq.(8) reads as

$$\partial_t u' = \dot{\xi} a + (g-2)u' + \mu^2 \partial_x^2 u' + \alpha h(x' + \xi), \qquad (9)$$

where $u'(x') = u(x)$, $[\partial_x \varrho^*(x)][\partial_x U_0(x)]/\varrho^*(x) = h(x' + \xi)$, $\partial_x U^*(x') = a(x')$ and $g(x') = (df/dU)(U^*(x')) + 2$. From $U(\xi(t),t) = U_0(\xi(t),t) = 0$ we get $u'(x' = 0, t) = 0$, which yields

$$-\dot{\xi} a(0) = \mu^2 \partial_x^2 u'(0,t) + \alpha h(\xi). \qquad (10)$$

Now (9) and (10) describe the time development of the velocity field in terms of u' and ξ.

Fourier transforming (9) and (10) we find that the short wavelength fluctuations ($k \to \infty$) are stabilized by the Laplacian term in (9). However, the growth rate λ of the long wavelength fluctuations, defined by $\xi \sim \exp(\lambda t)$, is determined by the characteristic equation

$$(-2 + 4\mu^2 - \lambda)\left(\frac{\alpha v_0}{\sqrt{2} D \mu} - \lambda\right) - \alpha \frac{10\sqrt{2} v_0 \mu}{D} = 0. \qquad (11)$$

It can be seen that (11) has positive solution for large enough α. This result means that for $\alpha > \alpha_c$ ($\alpha_c \approx 2\sqrt{2} D\mu/v_0$) the domain wall solution U^* is unstable, hence in this regime the walls disappear and all particles move in the same direction, *demonstrating that (7) and (8) predict an ordered phase in one dimension.*

To further confirm the relevance of the continuum theory to the discrete model (1), we integrate numerically (7) and (8). The results of the integration are in excellent agreement with the results obtained for the discrete model and with the linear stability analysis, and can be summarized as follows: (i) For the noiseless case ($\sigma = 0$) we find that for $\alpha > \alpha_c$ there is an ordered phase, which disappear for $\alpha < \alpha_c$; (ii) The ordered phase predicted by (11) is present for the noisy $\sigma > 0$ as well; (iii) Increasing σ leads to a second order phase transition from the ordered to the disordered state. Since σ plays the role of η in (1), this transition is equivalent with (3) observed in the discrete model; (iv) Finally, we measured the order parameter ϕ as a function of σ for various values of D. As Fig 2d. illustrates, for small D Eq.(3) with $\beta = 0.6$ provides an excellent fit to the numerical results, indicating that the discrete model (1) and the continuum theory (7) and (8) *belong to the same universality class.*

In conclusion, we showed that the SPP model exhibits spontaneous symmetry breaking and ordering in one dimension, a result surprising both in the light of the equilibrium spin models, and of the continuum theory investigated by TT. We introduced a new continuum theory, whose terms are explicitelly derived from the ingredients of the discrete model (in contrast with constructing it from symmetry arguments). Linear stability analysis indicates that an ordered phase can develop as the domain walls become unstable. While here we limited ourselves to the 1D case of the continuum theory, (7) and (8) can be generalized to arbitrary dimensions, thus apply to the physically and biologically relevant two and three dimensional models as well. Since the continuum model investigated by us contains the major ingredients of the SPP models, such as self-propulsion and local reorientation of the velocity, we expect our results to apply to a wide variety of systems made of SPP particles [1,2,5,7,8].

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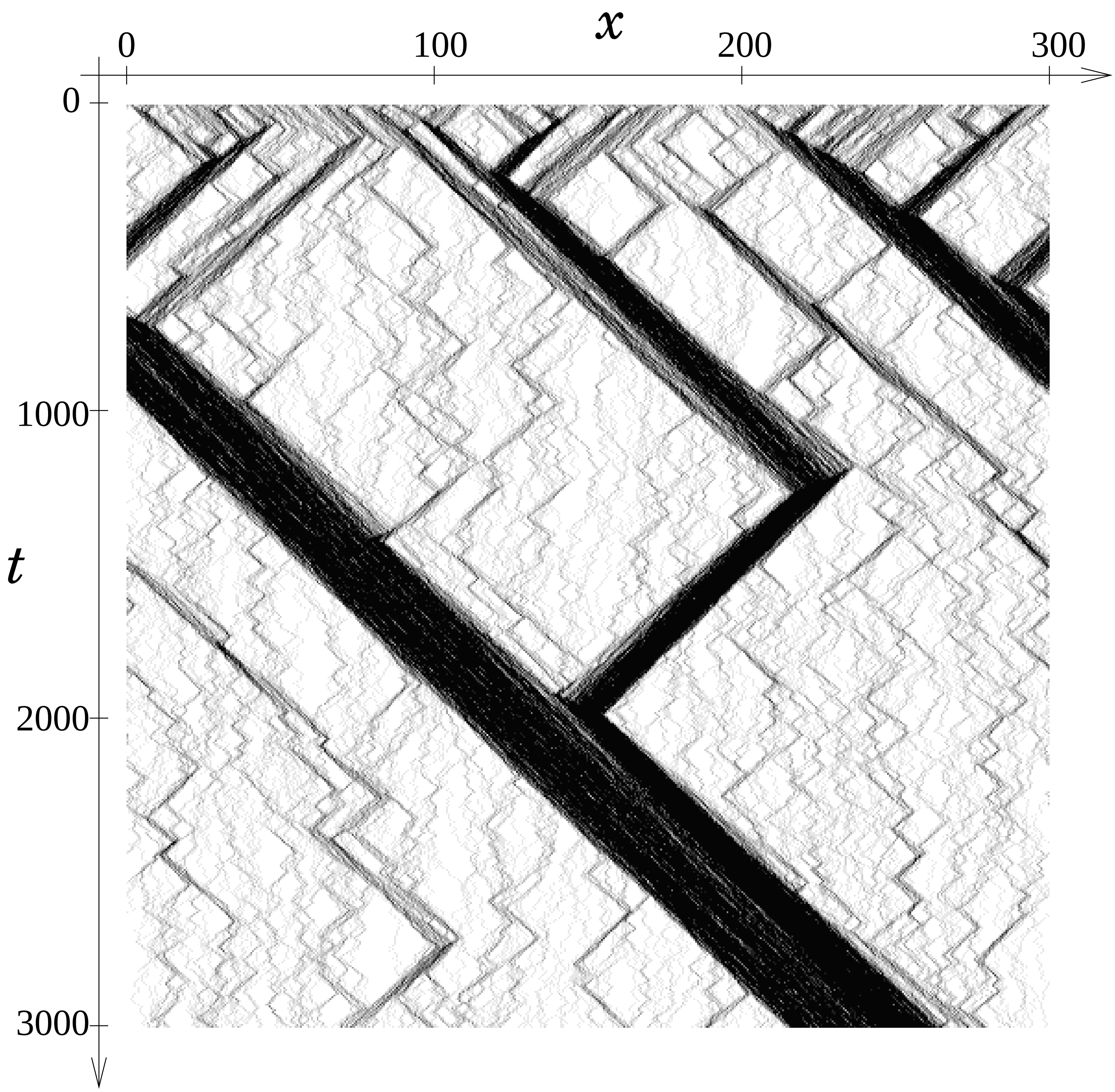
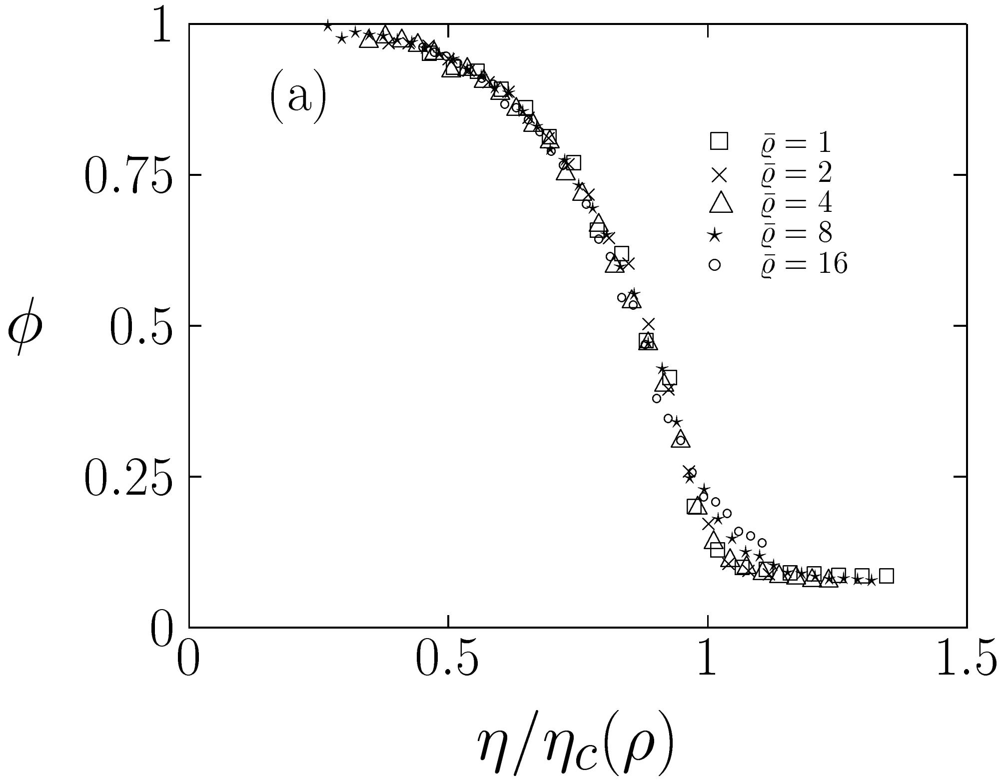
FIG. 1. The dynamics of the 1D SPP model for $L = 300$, $\eta = 2.0$ and $N = 600$. The darker gray level represents higher particle density. Note that the particles exhibit clustering and the spontaneous broken symmetry of motion.

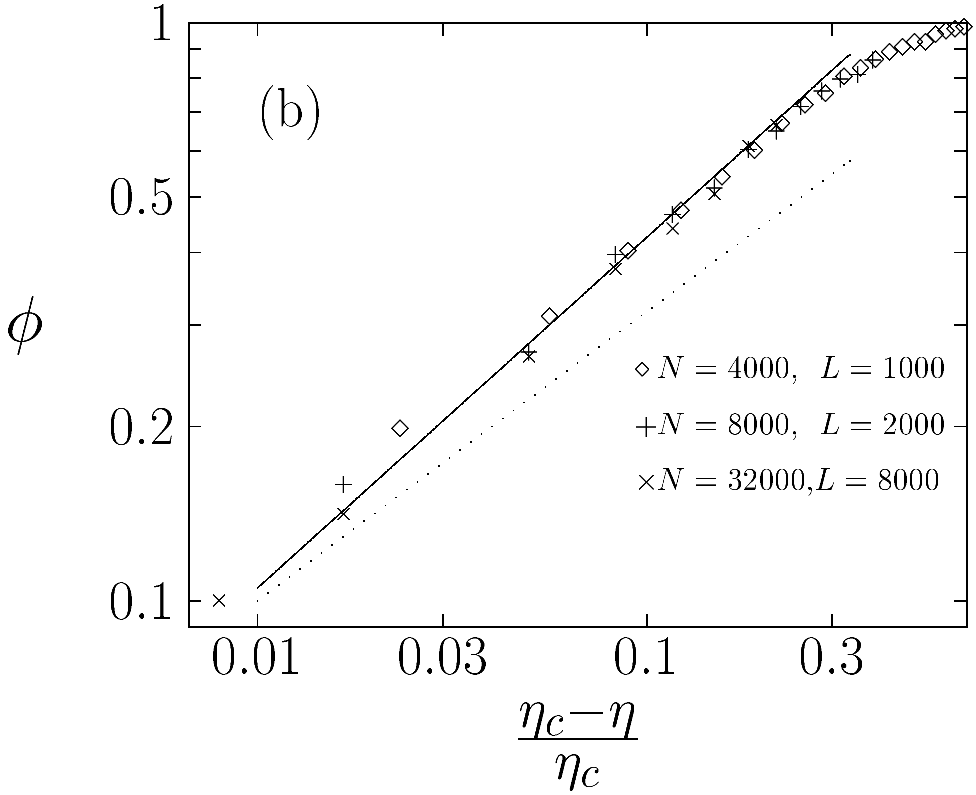
FIG. 2. a: The order parameter ϕ vs the noise amplitude normalized by the critical amplitude $\eta_c(\rho)$, for $L = 1000$ and various values of ρ. For $\eta < \eta_c(\rho)$ the system is in a symmetry-broken state indicated by $\phi > 0$. b: ϕ vanishes as a power-law in the vicinity of $\eta_c(\rho)$. Note the increasing scaling regime with increasing L. The solid line is a power-law fit with an exponent $\beta = 0.6$, while the dotted line shows the mean-field slope $\beta = 1/2$ as a comparison. c: Phase diagram in the $\rho - \eta$ plane. The critical line follows $\eta_c(\rho) \sim \rho^\kappa$. The solid curve represents a fit with $\kappa = 1/4$. d: The order parameter vs the std deviation of the noise normalized by $\sigma^* = \sigma_c(D = 0)$, obtained by direct numerical integration of the continuum model for $\alpha = 2$, $\mu = 1$, $v_0 = 0.1$, $\rho = 1$, $L = 1000$ and various values of D. For $D \ll 1$ $\phi(\sigma)$ follows a power-law with an exponent $\beta = 0.6$ (solid line).



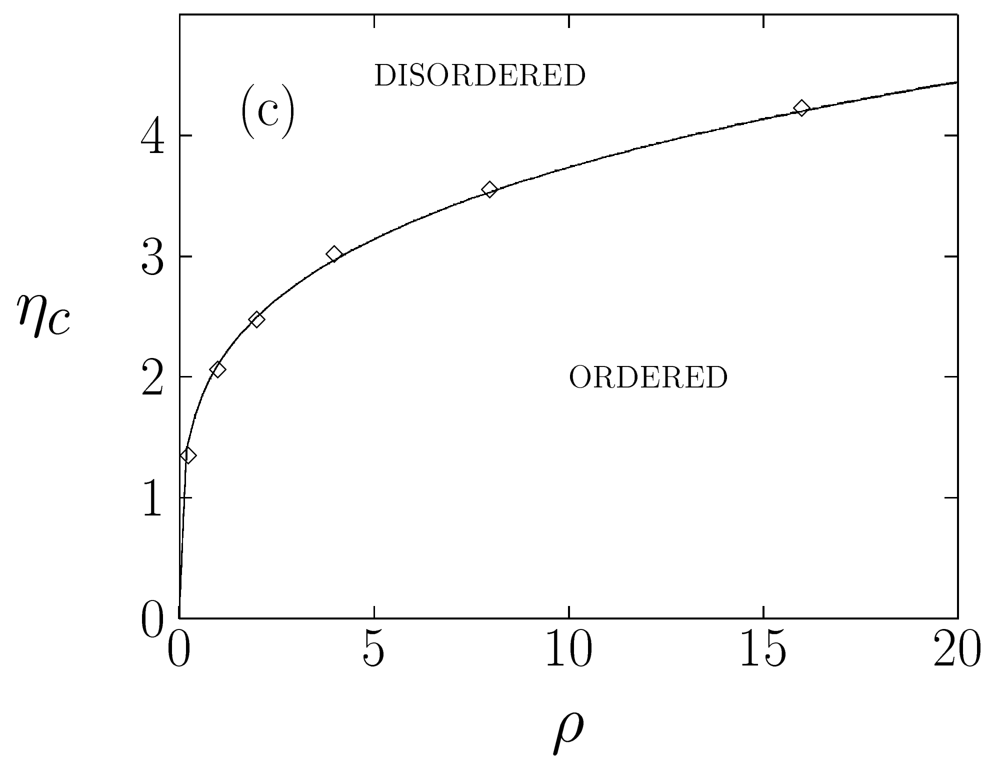
x
0
100
200
300
t
0
1000
2000
3000

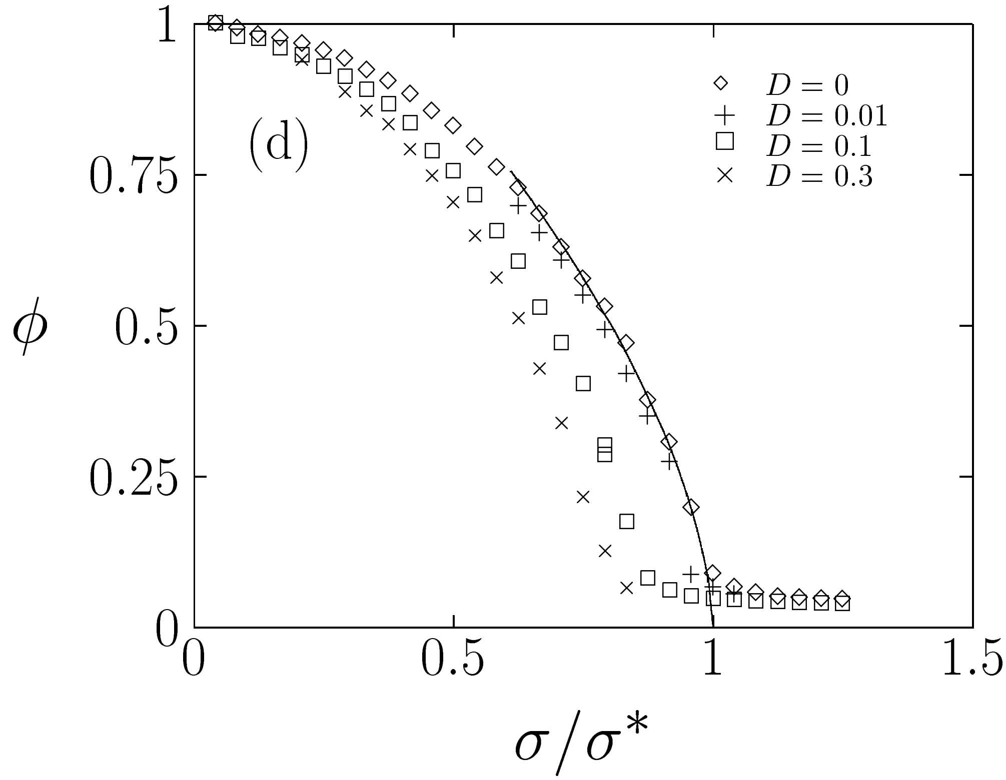


(a)
$\bar{\varrho} = 1$
$\bar{\varrho} = 2$
$\bar{\varrho} = 4$
$\bar{\varrho} = 8$
$\bar{\varrho} = 16$
ϕ
$\eta/\eta_c(\rho)$
0
0.25
0.5
0.75
1
0
0.5
1
1.5



(b)
ϕ
$\frac{\eta_c - \eta}{\eta_c}$
1
0.5
0.2
0.1
0.01
0.03
0.1
0.3
$N = 4000, \ L = 1000$
$N = 8000, \ L = 2000$
$N = 32000, L = 8000$


(c)
DISORDERED
ORDERED
η_c
ρ
0
1
2
3
4
0
5
10
15
20



(d)
$D = 0$
$D = 0.01$
$D = 0.1$
$D = 0.3$
ϕ
σ/σ^*
0
0.25
0.5
0.75
1
0
0.5
1
1.5